Optimal Enforcement Policies (Crackdowns) on a Drug Market

Kort, P.M.; Feichtinger, G.; Hartl, R.F.; Haunschmied, J.L.

Publication date:
1996

Optimal Enforcement Policies (Crackdowns) on a Drug Market

Peter M. Kort*, Gustav Feichtinger**, Richard F. Hartl***, Josef L. Haunschmied**

Abstract

In this paper an optimal control model is presented to design enforcement programs minimizing the social costs from both the market and crackdown. By using the maximum principle we show that performing an enforcement policy that leads to a collapse of the drug market is more likely to be optimal when the sales volume depends on the number of dealers. In case of a buyer's market the optimal enforcement policy leads to a saddle point equilibrium where the enforcement rate is fixed such that the number of dealers is kept constant at a positive level.

* Dept. of Econometrics and CentER, Tilburg University

** Dept. of OR and Systems Theory, Technical University of Vienna

*** Institute of Management Science, University of Vienna

1 Introduction

Illicit drug markets impose considerable costs on society, as do drug control efforts. In recent years problems devoted to drug policy have been increasingly studied in operations research and management science. In particular mathematical models have been presented to support the tactical question of optimal use of resources for a crackdown on a drug market.

In addition to source country control, interdiction, and high-level domestic enforcement, recent years have witnessed increasing attention to local drug enforcement (cf. Caulkins, 1990, 1993). In particular, crackdowns are being promoted as an efficient approach to drug control. Kleimann (1988) defines crackdowns as "*an intensive local enforcement effort directed at a particular geographic target*".

According to Caulkins (1993) crackdowns should be distinguished from the daily usual enforcement operations, which generally spread resources more or less uniformly. There is no consensus on the efficacy of crackdowns (cf. the discussion in Caulkins 1990, 1993). Mathematical models are formulated and analysed to describe how a drug market might respond to law enforcement.

Recently, the question of determining the optimal rate of enforcement pressure on a street-market for illicit drugs has been dealt with. In particular Baveja et al. (1992) analyse enforcement programs of finite duration that minimize the total costs of crackdown, subject to the constraint that the market is eliminated at the end of the program. The interesting analysis done by these authors is in the context of Caulkins' model (1990, see section 2). Their main result is that the simple strategy of using maximum available enforcement level until the market has collapsed is optimal in most instances (e.g. in the sellers' market). A drawback of the analysis in Baveja et al. (1992) is that artificial upper and lower boundaries are imposed upon the enforcement level. Since their optimal policy turns out to be bang bang, these exogenous boundaries are very important for the solution.

The purpose of the present paper is to extend the analysis of Baveja et al. (1992) in various directions. *First*, besides minimizing the costs of enforcement, we also include the current disutility (social costs) caused by the drugs market. It seems reasonable to represent the latter by the total number of dealers. *Second*, we consider an infinite planning period, and, *third*, the enforcement level is non negative and not bounded from above. Finally, we assume dealers to be risk seeking. It is shown that this latter assumption leads to a more gradual optimal enforcement policy, contrary to the bang bang behavior obtained in Baveja et al. (1992).

The paper is organized as follows. In Section 2 the model is presented. Section 3 states the necessary and sufficient optimality conditions resulting in a two-dimensional system of nonlinear differential equations. This system is studied more thoroughly in the Appendix. In Section 4 we find interesting results on the qualitative behaviour of optimal enforcement rates for three different scenarios. Finally, in Section 5 we draw some conclusions and give hints for possible extensions.

2 The Model

Since the state equation for the number of dealers is invented by Caulkins (1990), we briefly sketch his framework. His core assumption is that the rate of change of dealers depends on several market parameters as well as on the enforcement level of the police. These assumptions are similar to those commonly made in microeconomics. The dealers behave analogously to firms and the markets to industries. The drug markets are made up of a large number of identical dealers, and free entry and exit ensures zero long-run profits.

In the spirit of Becker (1976) dealers rationally maximize their utility. Thus, it seems plausible to assume that dealers enter the market as long as the utility available to dealers in the market exceeds their reservation wage, otherwise dealers will leave the market.

In particular, Caulkins specifies the rate of change of dealers with respect to time in a given market as follows:

$$\frac{dN(t)}{dt} = c_1 \left[\frac{\pi Q(N(t))}{N(t)} - \left(\frac{E(t)}{N(t)} \right)^{\gamma} - w_0 \right], \tag{1}$$

where

$N(t)$	$=$	the number of dealers in the market
t	$=$	time[1]
$Q(N)$	$=$	αN^{β} = number of sales per unit time
α, β	$=$	demand parameters, where $\beta \in [0,1], \alpha > 0$ and α, β are constants
$E(t)$	$=$	enforcement effort associated with a crackdown at time t
c_1	$=$	speed of adjustment ($c_1 > 0$ and constant)
π	$=$	generalized profit per transaction ($\pi > 0$ and constant)
w_0	$=$	a dealer's reservation wage ($w_0 > 0$ and constant)
γ	$=$	parameter associated with per dealer cost of enforcement effort ($\gamma \in (0,1)$ and constant).

Equation[1] (1) says that the rate of change of dealers is proportional to the difference between the utility of one dealer and his/her reservation wage (cf. Caulkins (1993), p. 852). The dealer's utility is his/her generalized profit (profit per sale net of pecuniary and other costs[2] times the number of sales per unit time, divided by the number of dealers) minus his/her risk from crackdown enforcement (proportional to the total enforcement per dealer). The reservation wage is what the dealer could earn in alternative employment including dealing elsewhere.

We omit the detailed discussion given by Caulkins (1993) to justify the power function for $Q(N)$ as well as costs imposed by the enforcement. Since dealers are identical and share the burden (risk) of enforcement equally, each dealer experiences costs being proportional to E/N. Caulkins assumes a power function to model the costs associated with crackdown, i.e. $(E/N)^{\gamma}$. He mainly considers the case of risk-averse dealers, i.e. $\gamma > 1$. But, as he remarks himself, dealers have selected a very risky profession so that it is not unreasonable to assume that they are risk-seeking. Therefore, in this paper we assume $\gamma < 1$. For risk seeking dealers the utility of income should be modelled as being convex, but it can be approximated as linear for small ranges. As can be inferred from equation

[1]For simplicity, in what follows we omit the time dependence in the variables, i.e. we write N for $N(t)$ etc.

[2]Caulkins (1993, p. 851) uses the term generalized profit instead of net profit because many of the costs are nonmonetary. The generalized profit equals the sales price minus the dealer's cost of doing business, including the costs imposed by other market participants and *conventional* police enforcement. The cost associated with *crackdowns* are dealt with separately.

(1) we use this approximation here (cf. Caulkins (1993), p. 852). From equation (1) we derive that the more risk seeking the dealers are (i.e. the lower the value of γ), the more enforcement is needed to shrink the market.

A dealer enters the market if and only if his/her utility exceeds that offered by other alternatives. The constant c_1 measures how fast dealers enter or leave the market.

We study the scenario where $\pi\alpha > w_0$, so that for N sufficiently low, and in case of no enforcement, the utility to be gained from this market is that large that dealers will enter.

Contrary to Caulkins (1993), whose model is descriptive, our aim is to determine the optimal rate of evolution of enforcement pressure. The objective is to minimize the discounted flow of social costs. These costs arise from putting resources on enforcement and from the disutility caused by the drugs market. If we represent the latter costs by the total number of dealers, the objective is given by

$$\text{maximize } \left\{ -\int_0^\infty C(E,N)\; e^{-rt} dt \right\},$$

where

$r =$ discount rate ($r > 0$ and constant).

For simplicity we assume a separable and linear cost function so that the objective becomes

$$\text{maximize } \left\{ -\int_0^\infty (E + \rho N) e^{-rt} dt \right\}, \tag{2}$$

where

$\rho =$ a positive constant measuring the relative cost of the drugs market.

3 Qualitative analysis in the phase plane

Define the current value Hamiltonian

$$H = -E - \rho N + \lambda c_1(\pi\alpha N^{\beta-1} - E^{\gamma}N^{-\gamma} - w_0), \tag{3}$$

where λ is the costate variable representing the shadow price of the number of dealers[3]. From (3) we derive the following necessary optimality conditions:

$$E = arg \max_E H,$$

which, due to the fact that dealers are assumed to be risk-seeking (i.e. $\gamma < 1$)[4], leads to

$$\lambda = -N^{\gamma}/c_1\gamma E^{\gamma-1}. \tag{4}$$

Furthermore, we have the following condition for the evolution of the costate variable

$$\dot{\lambda} = r\lambda - H_N = \rho + \lambda(r - c_1\pi\alpha(\beta-1)N^{\beta-2} - c_1\gamma E^{\gamma}N^{-\gamma-1}). \tag{5}$$

The plan is to solve the model by performing a phase plane analysis in the (N, E)-plane. Here we only state the main results. For a more thorough analysis the reader is referred to the Appendix. We first observe that the $\dot{N} = 0$ isocline immediately follows from (1):

$$E = (\pi\alpha N^{\beta-1} - w_0)^{1/\gamma}N \tag{6}$$

We conclude that on the $\dot{N} = 0$ isocline E will be positive for $N \in (0, N_{\max})$, where (the notation is borrowed from Caulkins (1993)):

$$N_{\max} = (\pi\alpha/w_0)^{1/(1-\beta)}. \tag{7}$$

In the Appendix we show that, provided that $\gamma + \beta > 1$, the maximum value of E on this isocline is reached for

[3]Note that concavity of H in N is not assured, so that sufficiency is hard to prove.

[4]Note that $H_{EE} < 0$, because $\gamma < 1$ and λ is negative.

$$N_{\min} = \left(\frac{\gamma+\beta-1}{\gamma}\right)^{\frac{1}{1-\beta}} \left(\frac{\pi\alpha}{w_0}\right)^{\frac{1}{1-\beta}} \tag{8}$$

(see Caulkins (1993, p. 857).

Next, we derive a differential equation for E. To do so we first differentiate (4) w.r.t. "t", which gives

$$\dot{\lambda} = \frac{N^{\gamma}}{c_1\gamma E^{\gamma-1}}\left[-\gamma N^{-1}\dot{N} + (\gamma-1)E^{-1}\dot{E}\right]. \tag{9}$$

After substitution of (4) and (9) into (5) and some rearranging, we obtain

$$\dot{E} = \frac{E}{\gamma-1}\left[\gamma N^{-1}\dot{N} + \rho c_1\gamma E^{\gamma-1}N^{-\gamma} - r + c_1\pi\alpha(\beta-1)N^{\beta-2} + c_1\gamma E^{\gamma}N^{-\gamma-1}\right]. \tag{10}$$

Substitution of (1) into (10) finally gives

$$\dot{E} = \frac{E}{\gamma-1}\left[\rho c_1\gamma E^{\gamma-1}N^{-\gamma} - r + c_1\pi\alpha(\gamma+\beta-1)N^{\beta-2} - c_1 w_0\gamma N^{-1}\right]. \tag{11}$$

In the Appendix we show that E must always be positive in equilibrium, so that we only need to consider the $\dot{E}=0$ isocline for positive E. From equation (11) we can obtain that this $\dot{E}=0$ isocline can be expressed as:

$$E = N \bigg/ \left[\frac{r}{\rho c_1\gamma}N - \frac{\pi\alpha}{\rho\gamma}(\gamma+\beta-1)N^{\beta-1} + \frac{w_0}{\rho}\right]^{\frac{1}{1-\gamma}}. \tag{12}$$

In the Appendix we prove that for the $\dot{E}=0$ isocline it holds that $E<0$ for $N\in(0,N_A)$ and $E>0$ for $N\in(N_A,\infty)$, where N_A satisfies

$$\frac{r}{c_1\gamma}N_A - \frac{\pi\alpha}{\gamma}(\gamma+\beta-1)N_A^{\beta-1} + w_0 = 0. \tag{13}$$

Concerning the status of the equilibrium the following proposition holds, from which the proof can be found in the Appendix.

Proposition 1

An equilibrium point is

- a saddle point, if in the equilibrium point it holds that $\left.\frac{dE}{dN}\right|_{\dot{E}=0} > \left.\frac{dE}{dN}\right|_{\dot{N}=0}$,
- unstable, if in the equilibrium point it holds that $\left.\frac{dE}{dN}\right|_{\dot{E}=0} < \left.\frac{dE}{dN}\right|_{\dot{N}=0}$.

4 The optimal enforcement policies

In this section we present optimal trajectories for different scenarios. We distinguish between a pure sellers' market (Subsection 4.1), a pure buyer's market (Subsection 4.2) and an intermediate case (Subsection 4.3).

4.1 Optimal trajectories in case of a pure sellers' market

An illicit drugs market is a pure sellers' market if it holds that each dealer creates its own demand. Here demand is abundant in the sense that if another dealer arrives, total market sales will expand enough that none of the existing dealers lose sales. In the model it means that the number of sales grow linearly with N so that $\beta = 1$.

To perform a phase plane analysis in the (N, E)-plane, we first obtain from (6) that for $\beta = 1$ the $\dot{N} = 0$ isocline is that straight line given by

$$E = (\pi\alpha - w_0)^{1/\gamma} N. \tag{14}$$

Concerning the $\dot{E} = 0$ isocline we conclude from Section 3 that $E < 0$ for $N \in (0, N_A)$, where N_A is defined by

$$N_A = (\pi\alpha - w_0)c_1\gamma/r. \tag{15}$$

It holds that $E > 0$ for $N \in (N_A, \infty)$. Furthermore, in the Appendix it is proved that $E(N_A^+) \to \infty$, and for $N > N_A$ the $\dot{E} = 0$ isocline is decreasing in the (N, E)-plane. The phase diagram in case of a pure sellers' market is depicted in Figure 1.

[Insert Figure 1 about here]

The equilibrium A in Figure 1 is unstable, which is confirmed by Proposition 1. This unstable equilibrium can be a node or a focus (see, e.g., Feichtinger and Hartl (1986), p. 105). If it is a focus, then we know from the literature (e.g. Skiba (1978) or Dechert

(1984)) that there exists an interval of N-values that contains the unstable equilibrium, and on which two candidate trajectories occur. One trajectory goes to the right and the other one goes to the left. The trajectory to the right lies on the $N-$axis, because the optimal trajectory is the one which is as far away from the $\dot{N} = 0$ isocline as possible (see Feichtinger and Hartl (1986), Theorems 4.13, 4.14). From Dechert (1984) we know that a so called Skiba-point, S say, can be identified such that for "large" initial numbers of drug dealers, $N > S$, the trajectory to the right is better, and for small initial numbers of drug dealers, $N < S$, the trajectory to the left generates a higher value of the objective.[5] These trajectories are called history dependent, since it depends on the history, i.e. on $N(0)$, which one is optimal. This situation is sketched in Figure 1.

It can also happen that the unstable equilibrium is a node. This means that we still have history dependent equilibria and the critical point (where to go to left or right) is simply the unstable node A. Since this situation is simpler than the case of a focus, we refrain from drawing a picture here.

To interpret the solution in Figure 1 it is convenient to write down the state equation for the number of dealers for $\beta = 1$ (cf. (1)):

$$\dot{N} = c_1 \left(\pi\alpha - \left(\frac{E}{N} \right)^{\gamma} - w_0 \right). \tag{16}$$

The utility of the individual dealer is $\pi\alpha - (E/N)^{\gamma}$. Each individual dealer, who enters the market, obtains a profit of $\pi\alpha$ from selling his drugs. This profit is independent from the total number of dealers that is already active on this market. We conclude that each dealer creates his own demand, which confirms that we have a sellers' market here.

We further conclude that enforcement effort does not have a large effect on the utility of the individual dealer if N is large. The reason is that the burden of enforcement is shared equally among the dealers, so that this burden is relatively low for the individual dealer when there are a lot of colleagues around. Then, according to (16), a large enforcement level does not prevent new dealers from entering the market so that the effectiveness of resources invested in enforcement is low. This makes it understandable that in Figure 1

[5] The reason is that for fixed N, the (maximized) Hamiltonian and therefore also the value function assumes its minimum value along the $\dot{N} = 0$ isocline so that on the left boundary of the overlapping interval the left trajectory is better, and on the right boundary the right trajectory is better; see also Feichtinger and Hartl (1986, p. 117). The existence of a Skiba point in between follows from continuity.

$E = 0$ for $N > S$.

For $N < S$ enforcement is effective enough so that it is optimal to invest resources in it. In fact, enforcement remains positive until the market collapses, i.e. the total number of dealers that operates on this market is zero in the end. This market collapse is a perfect illustration of the "positive feedback" generated by the enforcement (see, e.g., Kleiman (1988)). As enforcement increases, some dealers who are particularly sensitive to enforcement pressure exit the market. That increases the amount of enforcement per participant among those who remain, which might encourage still more to leave. The departure of this second group, even if total enforcement pressure remains the same, further increases the ratio of enforcement to the size of the market. In this way crackdowns provide a gain in efficiency which leads to a market collapse in this case.

4.2 Phase diagram in case of a pure buyer's market

A buyers' market is one in which sellers have little or no bargaining power. In the context of a drug market, we must think of a fixed number of sales with dealers simply fighting for market share; increasing the number of dealers would not increase the number of sales, because there is already a surplus of dealers. This can be modeled by setting $\beta = 0$.

Let us first look at the $\dot{N} = 0$ isocline for $\beta = 0$. From (6) we obtain that on the $\dot{N} = 0$ isocline E will be zero for $N = N_{\max}$ only (notice that here $\gamma + \beta < 1$ since $\beta = 0$ and $\gamma < 1$). Straightforward derivations give $\left.\frac{dE}{dN}\right|_{\dot{N}=0} < 0$ for $N \in [0, N_{\max})$, $\left.\frac{dE}{dN}\right|_{\dot{N}=0} = 0$ for $N = N_{\max}$ and $\left.\frac{d^2E}{dN^2}\right|_{\dot{N}=0} > 0$ for $N \in [0, N_{\max})$ (see also the Appendix).

Next, consider the $\dot{E} = 0$ isocline. According to (12) it looks as follows when $\beta = 0$:

$$E = N \Bigg/ \left[\frac{r}{\rho c_1 \gamma} N + \frac{(1-\gamma)\pi\alpha}{\rho\gamma} N^{-1} + \frac{w_0}{\rho}\right]^{\frac{1}{1-\gamma}}. \tag{17}$$

It is easy to see that on this isocline it holds that $E > 0$ when $N > 0$. If, while taking into account that $\beta = 0$, we put the part between brackets in (11) equal to zero, then we can obtain from this equation that

$$\left.\frac{dE}{dN}\right|_{\dot{E}=0, E>0} = \left[-\frac{rN^2}{c_1(1-\gamma)} + w_0 N + \frac{\pi\alpha(2-\gamma)}{\gamma}\right] \frac{N^{\gamma-3}E^{2-\gamma}}{\rho}. \tag{18}$$

After some algebra it turns out that $\left.\frac{dE}{dN}\right|_{\dot{E}=0,E>0} > 0$ for $N \in (0, \bar{N})$ and $\left.\frac{dE}{dN}\right|_{\dot{E}=0,E>0} < 0$ for $N \in (\bar{N}, \infty)$, where $\bar{N}$ is given by

$$\bar{N} = \frac{c_1(1-\gamma)}{2r}\left[w_0 + \left(w_0^2 + \frac{4r\pi\alpha(2-\gamma)}{c_1\gamma(1-\gamma)}\right)^{\frac{1}{2}}\right].$$

After substitution of $N = N_{\max}$ (cf. (7)) into (17) we get that for $E > 0$:

$$\left.\frac{dE}{dN}\right|_{\dot{E}=0,N=N_{\max}} = \left[\frac{-r(\pi\alpha)^2}{c_1(1-\gamma)w_0^2} + \frac{2\pi\alpha}{\gamma}\right]\frac{N_{\max}^{\gamma-3}E^{2-\gamma}}{\rho}. \tag{19}$$

From (19) we obtain that $\left.\frac{dE}{dN}\right|_{\dot{E}=0} > 0$ for $N = N_{\max}$, and thus that $N_{\max} < \bar{N}$ which implies that $\left.\frac{dE}{dN}\right|_{\dot{E}=0} > 0$ for $N \in (0, N_{\max}]$, if

$$\gamma < \frac{2c_1w_0^2}{2c_1w_0^2 + r\pi\alpha}. \tag{20}$$

The phase diagram for $\beta = 0$ and under condition (20) is depicted in Figure 2. According to Proposition 1 equilibrium A corresponds to a saddle point.

(Insert Figure 2 about here.)

To interpret this solution it again helps to write down the state equation for the number of dealers that holds in this case (cf. (1)):

$$\dot{N} = c_1\left(\frac{\pi\alpha}{N} - \left(\frac{E}{N}\right)^{\gamma} - w_0\right). \tag{21}$$

Here the utility of the individual dealer is $\pi\alpha/N - (E/N)^{\gamma}$. Total revenue from drug sales is fixed on this market and equals $\pi\alpha$. Since all dealers are identical, each of them gets an equal share of this revenue. Of course this share is low if the total number of dealers is large.

Contrary to the pure sellers' market case, here it is optimal to have positive enforcement for N large, despite of the fact that the burden of enforcement for the individual dealer is then low. The reason is that here it is possible to reduce the number of dealers for N large. This is because on a buyer's market with a large number of dealers the revenue per dealer is very low, and this negatively affects the change in the number of dealers per unit of time. Since $\gamma < 1$, the revenue per dealer is decreased more by a large N than the burden of the crackdown felt by the individual dealer. So, here enforcement pressure is relatively successful at driving away dealers, because a buyer's market with a large number of dealers is relatively unappealing to the dealers.

From Figure 2 we obtain that a market collapse will not occur. This is because when the number of dealers decreases the revenue per dealer increases more than the disutility from the enforcement pressure. Since $\gamma < 1$, dealers are risk seeking, which implies that the utility gained from large sales is larger than the disutility caused by the possibility of being arrested. Therefore, a market collapse requires an enormous amount of enforcement effort, since, due to the high revenue per dealer, it is very attractive for dealers to enter this market if N is low. This revenue effect more than offsets the positive feedback effect of the crackdown which played such a crucial role in the solution of the sellers' market. To obtain this result the assumption of dealers being risk seeking apparently is crucial. This is confirmed by Caulkins (1993) where it is shown that a buyer's market can collapse while enforcement effort is still finite. However, Caulkins also found that enforcement effort required to collapse a market is decreasing in β (see Caulkins (1993), Figure 3).

4.3 Phase diagram in the intermediate case

Here we study the case where the market is neither a sellers' market nor a buyer's market. In order to be able to say something about the $\dot{E} = 0$ isocline we assume that dealers are not too risk seeking, i.e. γ is close to 1. If we further assume that β is sufficiently large such that $2\gamma + \beta \geq 2$, we know from the Appendix that the $\dot{E} = 0$ isocline decreases in the (N, E)-plane.

From Section 3 we get that for $\beta < 1$ and $2\gamma + \beta \geq 2$ the $\dot{N} = 0$ isocline reaches the highest value for E when $N = N_{\min}$ (see eqn. (8)). Furthermore, we know that on the $\dot{N} = 0$ isocline E will be zero for $N = 0$ and for $N = N_{\max}$ (see eqn. (6)), and E is positive for $N \in (0, N_{\max})$. Information about first and second order derivatives can be found in the Appendix.

We conclude that the following inequality is a sufficient (but not necessary!) condition for the $\dot{E} = 0$ isocline to intersect the $\dot{N} = 0$ isocline twice:

$$E(N_{\min})\,|_{\dot{E}=0} < E(N_{\min})|_{\dot{N}=0}\,, \tag{22}$$

which, by using (6), (8), (12) and doing some calculations, can be rewritten into

$$N_{\min}^{\frac{1}{\gamma-1}} = \left(\frac{\gamma w_0}{(\gamma+\beta-1)\pi\alpha}\right)^{\frac{1}{(1-\beta)(1-\gamma)}} < \left(\frac{r}{\rho c_1 \gamma}\right)^{\frac{1}{1-\gamma}} \left(\frac{(1-\beta)w_0}{\gamma+\beta-1}\right)^{\frac{1}{\gamma}}. \tag{23}$$

If (23) does not hold it is still possible that the isoclines intersect twice, but it can also happen that there is only one intersection (hairline case!) or no intersection at all. Here, we restrict ourselves to the case where (23) holds so that we are sure of the existence of two equilibria. The phase diagram is depicted in Figure 3.

(Insert Figure 3 about here.)

Due to Proposition 1 we can conclude that A is a saddle point, while B is unstable. Here a Skiba point N_S exists such that for $N(0) > N_S$ the equilibrium A will be reached, while for $N(0) < N_S$ we will have a market collapse in the long run.

Figure 3 is drawn for the case where the unstable equilibrium B is a focus, but, like in Figure 1, it could also be a node. Then for $N > B$ it is optimal to converge to A, while for $N < B$ it is optimal to approach the origin. We conclude that in both cases (i.e. node or focus) the equilibrium A is history dependent, because it depends on the history, i.e. $N(0)$, whether A or the origin will be reached in the long run. This solution is really intermediate in the sense that it has the possibility of market collapse from the sellers' market solution and the occurrence of the stable equilibrium from the buyer's market solution.

Figure 3 relates to the (descriptive) analysis of the dynamic equation (1) by Caulkins (1993), in which he considered E to be constant. Nevertheless some of his interpretations carry over to our solution of Figure 3, in which E is not kept constant but optimally determined instead. As argued by Caulkins (1993), if there were very few dealers each would attract considerable enforcement pressure. So it would be unprofitable to deal, and dealers would exit. Suppose, on the other hand, that there were an extremely large number of dealers. Then none would suffer terribly from enforcement, but there would not be enough customers to go around. Again dealers would exit. There may be intermediate values of N, however, that are large enough to dilute enforcement pressure but small enough for each dealer to make a reasonable number of sales. In that range, dealers' return would exceed the reservation wage, so more dealers would enter. Apparently, making enforcement pressure so large that the market would collapse in the end is too expensive for being optimal.

5 Concluding Remarks

In this paper we dealt with the question how police should design the rate at which to crackdown on a market for illicit drugs in order to minimize the social cost over time.

Drug dealers are modeled analogously to profit seeking firms. Considering the number of dealers on the market as state variable its rate of change is assumed to be proportional to the difference between the utility available to one dealer and his/her reservation wage. Dealers enter the market as soon as the profit from drug sales minus the threat of being caught from enforcement exceeds the wage the dealer could earn in alternative employment. This nonlinear state equation has been extensively studied by Caulkins (1990, 1993) and Baveja et al. (1993). Different to Baveja et al. (1992) we assume risk-seeking dealers guaranteeing that a sufficiency condition of the maximum principle (the Legendre-Clebsch) condition) is satisfied.

Solutions were obtained for three different scenarios: a sellers' market, a buyer's market and the intermediate case. For a sellers' market it is optimal to have positive enforcement if the size of the market is sufficiently small. Due to the increasing efficiency of enforcement with decreasing market size this leads eventually to a market collapse. For a buyer's market, however, this increasing efficiency effect is more than offset by the fact that sales are large in case the number of dealers is low. Therefore optimal enforcement effort leads to a stable equilibrium with a positive number of dealers rather than to a market collapse. Here, it is important to remark that this outcome results from the assumption that dealers are risk seeking. For the intermediate scenario we have a market collapse in case the initial size of the market is sufficiently small. For higher initial sizes the number of dealers converges to a stable saddle point equilibrium.

Concerning future research several interesting extensions can be considered. First, as already observed by Caulkins (1993) some crackdowns explicitly seek to arrest users. Therefore total sales on the market should also decrease with the enforcement rate, rather than that they only increase with the number of dealers as it is modeled now.

Second, in reality social cost of a drugs market is more than proportionally increasing with the size. Therefore our linear objective should be made convex in the number of dealers.

Third, it is interesting to find out how the optimal enforcement policy looks like in case of risk averse dealers. Within the present model formulation the assumption of dealers being risk averse would lead to an uninterpretable chattering control policy. This can be circumvented by introducing the enforcement rate as a second state variable in the model and imposing convex adjustment costs on the rate of change of enforcement. It is well known that a model created like this can generate stable limit cycles (see, e.g., Feichtinger, Novak and Wirl (1994)).

A fourth extension would be the explicit inclusion of the number of addicts as a second state variable. Dealers and addicts are in a symbiotic relation, and some interesting results might be expected. One problem would be how to allocate a budget between crackdown and therapy. Clearly, then the number of addicts would appear as a third term in the objective functional.

Acknowledgement. The help of E. van Damme, H. Dawid, E.J. Dockner, A. van den Elzen, A. Novak and F. Wirl is gratefully acknowledged.

Appendix. Mathematical Analysis of the Dynamic System

First we derive under what condition an equilibrium corresponds to a saddle point. Then we study the $\dot{N}=0$ isocline and $\dot{E}=0$ isocline thoroughly.

A.1 Condition for existence of a saddle point.

The determinant of the Jacobian of the dynamic system ((1),(11)) evaluated at the equilibrium equals:

$$\begin{aligned}\det J = \tfrac{c_1^2}{1-\gamma} \quad & \Big[\pi\alpha\gamma(1-\beta)E^{\gamma-1}N^{\beta-\gamma-3}\left\{\rho\gamma N+(\gamma+\beta)E\right\}+ \\ & (1-\beta)\pi\alpha N^{\beta-3}\left\{-\tfrac{r}{c_1}N-\pi\alpha(1-\beta)N^{\beta-1}\right\}+\tfrac{r}{c_1}\gamma E^{\gamma}N^{-\gamma-1}\Big].\end{aligned} \tag{A.1}$$

From (A.1) we obtain that this determinant contains positive and negative terms. Hence, it depends on the specific location of an equilibrium in the (N,E)-plane whether it is a saddle point or not.

Notice that at a steady state $E=0$ can never hold. The reason is that $E=0$ implies that $\lambda=0$ (see eqn. (4)), and at a steady state $\lambda=0$ cannot hold, because $\lambda=0$ in turn implies that $\dot{\lambda}=\rho$, due to (5). Therefore, E must always be positive in equilibrium. According to (11) this implies that

$$\rho c_1\gamma E^{\gamma-1}N^{-\gamma}-r+c_1\pi\alpha(\gamma+\beta-1)N^{\beta-2}-c_1 w_0\gamma N^{-1}=0. \tag{A.2}$$

Substitution of this expression into (A.1), as well as using that $\dot{N}=0$ in an equilibrium (cf. (1)), leads to

$$\begin{aligned}\det J = \tfrac{E^{\gamma-1}N^{-\gamma-2}\gamma c_1^2}{1-\gamma} \quad & \Big[(1-\beta)\pi\alpha N^{\beta-1}\left\{(\gamma-1)\rho N+(\gamma+\beta-2)E\right\}+ \\ & +\gamma E^{\gamma}N^{-\gamma}(E+\rho N)\Big].\end{aligned} \tag{A.3}$$

From equation (1) we obtain that

$$\left.\frac{dE}{dN}\right|_{\dot{N}=0}=\frac{(\beta-1)\pi\alpha E^{-\gamma+1}N^{\beta+\gamma-2}+\gamma E N^{-1}}{\gamma}. \tag{A.4}$$

Furthermore, from (A.2) it can be obtained that

$$\left.\frac{dE}{dN}\right|_{\dot{E}=0,E>0} = \frac{-\gamma E N^{-2}(E+\rho\gamma N) - \pi\alpha(1-\beta)(\gamma+\beta-2)E^{-\gamma+2}N^{\beta+\gamma-3}}{\rho\gamma(1-\gamma)}. \quad \text{(A.5)}$$

Now, we are ready to prove Proposition 1, which is started here once more.

Proposition 1.
An equilibrium point is

- a saddle point, if in the equilibrium point it holds that $\left.\frac{dE}{dN}\right|_{\dot{E}=0} > \left.\frac{dE}{dN}\right|_{\dot{N}=0}$,
- unstable, if in the equilibrium point it holds that $\left.\frac{dE}{dN}\right|_{\dot{E}=0} < \left.\frac{dE}{dN}\right|_{\dot{N}=0}$.

Proof
From (A.4) and (A.5) we obtain

$$\begin{aligned}\left.\frac{dE}{dN}\right|_{\dot{N}=0} - \left.\frac{dE}{dN}\right|_{\dot{E}=0} = \frac{1}{\rho\gamma(\gamma-1)} \quad & \left[\pi\alpha N^{\beta-1}E^{1-\gamma}N^{\gamma-2}\left\{(\beta-1)(\gamma-1)\rho N +\right.\right. \\ & \left. + (\beta-1)(\gamma+\beta-2)E\right\} + \\ & \left. -\gamma E N^{-2}\left\{-\rho(\gamma-1)N + E + \rho\gamma N\right\}\right],\end{aligned}$$

which can be rewritten into

$$\begin{aligned}\left.\frac{dE}{dN}\right|_{\dot{N}=0} - \left.\frac{dE}{dN}\right|_{\dot{E}=0} = \frac{E^{1-\gamma}N^{\gamma-2}}{\rho\gamma(\gamma-1)} \quad & \left[\pi\alpha N^{\beta-1}(\beta-1)\left\{(\gamma-1)\rho N + (\gamma+\beta-2)E\right\} + \right. \\ & \left. -\gamma E^{\gamma}N^{-\gamma}(E+\rho N)\right].\end{aligned} \quad \text{(A.6)}$$

Substitution of (A.3) into (A.6) gives

$$\left.\frac{dE}{dN}\right|_{\dot{N}=0} - \left.\frac{dE}{dN}\right|_{\dot{E}=0} = \frac{E^{2-2\gamma}N^{2\gamma}}{\rho c_1^2\gamma^2}\det J. \quad \text{(A.7)}$$

From (A.7) we obtain that

$$- \text{ saddlepoint } \Rightarrow \det J < 0 \Rightarrow \left.\frac{dE}{dN}\right|_{\dot{E}=0} > \left.\frac{dE}{dN}\right|_{\dot{N}=0}, \quad \text{(A.8)}$$

$$- \text{ unstable } \Rightarrow \det J > 0 \Rightarrow \left.\frac{dE}{dN}\right|_{\dot{E}=0} < \left.\frac{dE}{dN}\right|_{\dot{N}=0}. \quad \text{(A.9)}$$

Q.e.d.

A.2 A closer look at the $\dot{N} = 0$ isocline.

To find maximum or minimum values of E on the $\dot{N} = 0$ isocline, we differentiate (6) w.r.t. N:

$$\frac{dE}{dN} = \left\{ \frac{(\gamma + \beta - 1)}{\gamma} \pi \alpha N^{\beta - 1} - w_0 \right\} \left(\pi \alpha N^{\beta - 1} - w_0 \right)^{\frac{1}{\gamma} - 1}. \tag{A.10}$$

Hence, dE/dN will be zero for $N = N_{\max}$ (due to the fact that $\gamma < 1$) and, provided that $\gamma + \beta > 1$, for $N = N_{\min}$.

Comparing (7) and (8) we get that $N_{\max} = N_{\min}$ for $\beta = 1$. Notice also that via (A.1) we know that $\det J$ is positive for $\beta = 1$, so that we have unstability in this case. For $\beta < 1$ we have that $N_{\min} < N_{\max}$.

From (A.10) it is easy to obtain that

$$dE/dN > 0 \text{ for } N \in (0, N_{\min}), \tag{A.11a}$$

$$dE/dN < 0 \text{ for } N \in (N_{\min}, N_{\max}). \tag{A.11b}$$

To obtain some more information about the shape of the $\dot{N} = 0$ isocline, we also calculate the second order derivative:

$$\frac{d^2E}{dN^2} = -\left(\pi \alpha N^{\beta - 1} - w_0 \right)^{\frac{1}{\gamma} - 2} \frac{(1 - \beta) \pi \alpha N^{\beta - 2}}{\gamma} \left[\frac{(\gamma + \beta - 1)}{\gamma} \pi \alpha N^{\beta - 1} - w_0 \beta \right] \tag{A.12}$$

We conclude that on the interval $(0, N_{\max})$ the second order derivative changes sign for that N, say $\hat{N}$, that satisfies

$$\hat{N} = \frac{1}{\beta^{1/(1-\beta)}} \left(\frac{(\gamma + \beta - 1) \pi \alpha}{\gamma w_0} \right)^{\frac{1}{1-\beta}} = \frac{1}{\beta^{1/(1-\beta)}} N_{\min}. \tag{A.13}$$

Notice that $\hat{N}$ is only positive when $\gamma + \beta > 1$. If $\beta = 1$, then $\hat{N} = N_{\min} = N_{\max}$. For $\beta < 1$ it holds that $N_{\min} < \hat{N} < N_{\max}$. From (A.12) we further derive that

$$\frac{d^2E}{dN^2} < 0 \text{ for } N \in (0, \hat{N}), \tag{A.14}$$

$$\frac{d^2E}{dN^2} > 0 \text{ for } N \in (\hat{N}, N_{\max}). \tag{A.15}$$

A.3 A closer look at the $\dot{E} = 0$ isocline

Consider the denominator of (12). If $\gamma + \beta > 1$ it holds that the denominator is negative for $N \downarrow 0$. For $N = N_{\max}$ the denominator equals

$$\left[\frac{r}{\rho c_1 \gamma}\left(\frac{\pi\alpha}{w_0}\right)^{\frac{1}{1-\beta}} + (1-\beta)\frac{w_0}{\rho}\right]^{\frac{1}{1-\gamma}},$$

which is positive. Hence, by continuity it follows that for an $N \in (0, N_{\max})$, say N_A, it must hold that this denominator equals zero, so that N_A satisfies

$$\frac{r}{c_1\gamma}N_A - \frac{\pi\alpha}{\gamma}(\gamma+\beta-1)N_A^{\beta-1} + w_0 = 0. \tag{A.16}$$

We conclude that on the $\dot{E} = 0$ isocline it holds that $E < 0$ for $N \in (0, N_A)$ and $E > 0$ for $N \in (N_A, \infty)$. Since $E < 0$ makes no economic sense we do not need to consider the part of this isocline where $N \in (0, N_A)$ any further. Furthermore, it can be concluded that

$$\lim_{N \downarrow N_A} E(N_A)\Big|_{\dot{E}=0} = \infty. \tag{A.17}$$

Also, from (8) and (12) we obtain that

$$E(N_{\min})\,|_{\dot{E}=0} = \left(\frac{\rho c_1 \gamma}{r}\right)^{\frac{1}{1-\gamma}} N_{\min}^{\frac{\gamma}{\gamma-1}} > 0. \tag{A.18}$$

Hence, we can conclude that $N_{\min} > N_A$.

Next, we determine the derivative of the $\dot{E} = 0$ isocline in the (N, E)-plane. To do so we put the part between brackets in (11) equal to zero and obtain from this equation that

$$\frac{dE}{dN} = \left[-\frac{r}{\rho c_1(1-\gamma)}N^{\gamma-1} + \frac{(\gamma+\beta-1)(\gamma+\beta-2)}{\rho\gamma(1-\gamma)}\pi\alpha N^{\beta+\gamma-3} + \frac{w_0}{\rho}N^{\gamma-2}\right]E^{2-\gamma}. \tag{A.19}$$

After noticing that

$$(\gamma+\beta-1)(\gamma+\beta-2)=\gamma(\gamma-1)+(2\gamma+\beta-2)(\beta-1),$$

we see that we can rewrite (A.19) into

$$\frac{dE}{dN}=\left[-\frac{rN}{c_1(1-\gamma)}-\frac{(2\gamma+\beta-2)(1-\beta)}{\gamma(1-\gamma)}\pi\alpha N^{\beta-1}-\pi\alpha N^{\beta-1}+w_0\right]\frac{N^{\gamma-2}E^{2-\gamma}}{\rho} \tag{A.20}$$

From (7) we obtain that, since $\beta\in(0,1]$, we have

$$\pi\alpha N^{\beta-1}\geq w_0 \text{ for } N\in[0,N_{\max}], \tag{A.21}$$

so that from (A.21) we can conclude that for $N\in(N_A,N_{\max}]$ it holds that

$$\left.\frac{dE}{dN}\right|_{\dot{E}=0}<0 \tag{A.22}$$

if

$$2\gamma+\beta\geq 2. \tag{A.23}$$

References

1. Baveja, A., R. Batta, J.P. Caulkins and M.H. Karwan, "Collapsing street market for illicit drugs: the benefits of being decisive", Working Paper (1992).
2. Baveja, A., R. Batta, J.P. Caulkins and M.H. Karwan, "Modeling the response of illicit drug market to local enforcement", *Socio-Econ. Plann. Sci.* **27**, 73-89 (1993).
3. Becker, G.S., "*The Economic Approach to Human Behavior*", Univ. of Chicago Press, Chicago (1976).
4. Caulkins, J.P., "The distribution and consumption of illicit drugs: some mathematical models and their policy implications", unpubl. Ph.D. Dissertation, Massachusetts Inst. of Technology, Cambridge Mass. (1990).
5. Caulkins, J.P., "Thinking about displacement in drug markets: why observing change of venue isn't enough", *J. Drug Issues*, **22**, 17-30 (1992).
6. Caulkins, J.P., "Local drug markets' response to focused police enforcement", *Operations Research*, **41**, 848-863 (1993).
7. Caulkins, J.P. (Ed.), "Mathematical models of drug markets and drug policy", Special issue of *Mathematical and Computing Modelling*, **17**, 1-115 (1993).
8. Dawid, H. and G. Feichtinger, "Optimal allocation of drug control efforts: a differential game analysis", to appear in *Journal of Optimization Theory and Applications*, (1995).
9. Dechert, W.D., "Has the Averch-Johnson effect been theoretically justified", *Journal of Economic Dynamics and Control*, 8, 1-17 (1984).
10. Gragnani, A., Feichtinger, G. and S. Rinaldi, "Dynamics of drug consumption: a theoretical model", IIASA Working Paper (1994).
11. Feichtinger, G., Hartl, R.F., *"Optimale Kontrolle Oekonomischer Prozessc: An wendungen des Maximumprinzips in den Wirtschaftswissenschaften"*, DeGruyter, Berlin (1986).
12. Feichtinger, G., Hartl, R.F. and S.P. Sethi, "Dynamic optimal control models in advertising: recent developments", *Management Science*, **40**, 195-226 (1994).
13. Feichtinger, G. Novak, A. and F. Wirl, "Limit cycles in intertemporal adjustment models", *Journal of Economic Dynamics and Control*, **18**, 353-380 (1994).

14. Kleimann, M.A.R., "Crackdowns: The Effects of Intensive Enforcement on Retail Herion Dealing. In *"Street-Level Drug Enforcement: Examining the Issues"*, M.R. Chaiken (ed.), Natural Institute of Justice, Washington, D.C. (1988).

15. Kleimann, M.A.R., "Enforcement swamping: a positive feedback mechanism in rates of illicit activity", *Mathematical and Computing Modelling*, **17**, 65-76 (1993).

16. Skiba, A.K., "Optimal growth with a convex-concave production function", *Econometrica*, 46, 527-539 (1978).

Figure 1. Optimal trajectories in case of a pure sellers' market ($\beta = 1$).

Figure 2. Optimal trajectories in case of a pure buyer's market ($\beta = 0$) and (44).

Figure 3. Optimal trajectories under the conditions $2\gamma + \beta \geq 2$ and (47).